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Some viscous fluid cosmological models of plane symmetry

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Abstract. Two plane-symmetric cosmological models representing viscous fluid with free gravitational field of type D have been obtained. The effect of viscosity on various kinematical parameters has been discussed.

1. Introduction

Astronomical observations of the large-scale distribution of galaxies in our universe show that the distribution of matter can be satisfactorily described by a perfect fluid. It has, however, been conjectured that some time during an earlier phase in the evolution of the universe when galaxies were formed, the material distribution behaved like a viscous fluid (Ellis 1971, p 124). It is therefore of interest to obtain cosmological models for such distributions. It is also well known that there is a certain degree of anisotropy in the actual universe. We therefore choose the metric for the cosmological model to be plane-symmetric. In this paper we have obtained two models in which the material distribution is that of a viscous fluid. It is found that the kinematic viscosity prevents shear, expansion and the free gravitational field from withering away for large values of t.

2. The field equations

We consider the plane-symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \tag{2.1}$$

where A, B, C are functions of t alone. This ensures that the model is spatially homogeneous. The energy–momentum tensor for a viscous fluid distribution is given by (Landau and Lifshitz 1963)

$$T_i^k = (\epsilon+p)V_iV^k + pg_i^k - \eta(V_{i,}^k + V_{,i}^k + V^kV^lV_{i,l} + V_iV^lV_{,l}^k) - (\zeta - \tfrac{2}{3}\eta)V_{,l}^l(g_i^k + V_iV^k) \tag{2.2}$$

together with

$$V_iV^i = -1 \tag{2.3}$$

p being the isotropic pressure, ϵ the density, η and ζ the two coefficients of viscosity and a comma indicates covariant differentiation. V^i is the flow vector satisfying equation

(2.3). We assume the coordinates to be comoving so that $V^1=V^2=V^3=0$ and $V^4=1/A$. The field equations

$$R_i^k-\tfrac{1}{2}R\delta_i^k+\Lambda\delta_i^k=-8\pi T_i^k \tag{2.4}$$

for the line element (2.1) are as follows:

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B}-\frac{C_{44}}{C}-\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right)-\Lambda=8\pi\left[p-2\eta\frac{A_4}{A^2}-\left(\zeta-\frac{2}{3}\eta\right)V^l_{;l}\right] \tag{2.5}$$

$$\frac{1}{A^2}\left(-\frac{C_{44}}{C}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right)-\Lambda=8\pi\left[p-2\eta\frac{B_4}{AB}-\left(\zeta-\frac{2}{3}\eta\right)V^l_{;l}\right] \tag{2.6}$$

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right)-\Lambda=8\pi\left[p-2\eta\frac{C_4}{AC}-\left(\zeta-\frac{2}{3}\eta\right)V^l_{;l}\right] \tag{2.7}$$

and

$$\frac{1}{A^2}\left(\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right)+\Lambda=8\pi\epsilon. \tag{2.8}$$

The suffix 4 after the symbols A, B, C, denotes ordinary differentiation with respect to t. These are four equations in the five unknowns A, B, C, ϵ and p. The coefficients of viscosity η and ζ are taken as constants. Equations (2.5)–(2.8) are not independent, but are related by the contracted Bianchi identities. In the present case they lead to the single condition

$$\frac{d\epsilon}{dt}+(\epsilon+p)\frac{d}{dt}\ln(ABC)-\left(\rho-\frac{2}{3}\eta\right)\frac{1}{A}\left(\frac{d}{dt}\ln(ABC)\right)^2-\frac{2\eta}{A}\left(\frac{A_4^2}{A^2}+\frac{B_4^2}{B^2}+\frac{C_4^2}{C^2}\right)=0.$$

For complete solution of equations (2.5)–(2.8) we need an extra condition. An obvious one is the imposition of an equation of state. However we proceed from a different consideration. Although the distribution of matter at each point determines the nature of expansion in the model, the latter is also affected by the free gravitational field through its effect on the expansion, vorticity and shear in the fluid flow. A prescription of such a field may therefore be made on an *a priori* basis. The cosmological models of Robertson and Walker, as well as the universes of Einstein and De Sitter, have vanishing free gravitational fields. In this paper we choose the free gravitational field to be of type D which is of the next order in the hierarchy of Petrov classification. This requires that, either

(*a*) $C_{12}{}^{12}=C_{13}{}^{13}$

or

(*b*) $C_{12}{}^{12}=C_{23}{}^{23}$.

Conditions (*a*) and (*b*) are identically satisfied if $B=C$ and $A=C$ respectively. However, we shall assume A, B, C to be unequal on account of the supposed anisotropy.

From equations (2.5) and (2.6) we get

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{B_{44}}{B}-\frac{B_4C_4}{BC}=16\pi\eta A\left(\frac{B_4}{B}-\frac{A_4}{A}\right). \tag{2.9}$$

Also, from equations (2.6) and (2.7) we get

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=16\pi\eta A\left(\frac{C_4}{C}-\frac{B_4}{B}\right). \tag{2.10}$$

3. The first model

The condition $C_{12}{}^{12}=C_{13}{}^{13}$ leads to

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}+2\frac{A_4}{A}\left(\frac{C_4}{C}-\frac{B_4}{B}\right)=0. \tag{3.1}$$

Equations (2.10) and (3.1) lead to

$$A=\frac{1}{8\pi\eta t+a} \tag{3.2}$$

where a is a constant of integration. From equations (3.1) and (3.2) we get

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=-\frac{16\pi\eta}{(8\pi\eta t+a)}\left(\frac{B_4}{B}-\frac{C_4}{C}\right) \tag{3.3}$$

which on integration gives

$$B_4C-BC_4=\frac{b}{(8\pi\eta t+a)^2} \tag{3.4}$$

b being a constant of integration. From equations (2.9) and (3.2) we get

$$\left(\frac{8\pi\eta}{8\pi\eta t+a}\right)^2+\frac{8\pi\eta}{8\pi\eta t+a}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{16\pi\eta}{8\pi\eta t+a}\frac{B_4}{B}+\frac{B_{44}}{B}+\frac{B_4C_4}{BC}=0. \tag{3.5}$$

From equations (3.4) and (3.5) we obtain

$$B=\frac{K(Lt+M)^{\frac{1}{2}+\frac{1}{2}b/L}}{(8\pi\eta t+a)} \tag{3.6}$$

and

$$C=\frac{(Lt+M)^{\frac{1}{2}-\frac{1}{2}b/L}}{K(8\pi\eta t+a)} \tag{3.7}$$

where K, L and M are constants of integration. From equations (2.7) and (2.8) we get

$$8\pi p=-64\pi^2\eta^2+\frac{(L^2-b^2)(8\pi\eta t+a)^2}{4(Lt+M)^2}+\frac{32\pi\eta aL-128\pi^2\eta^2(Lt+3M)}{3(Lt+M)}$$
$$-8\pi\zeta\left(24\pi\eta-\frac{(8\pi\eta t+a)L}{(Lt+M)}\right)-\Lambda \tag{3.8}$$

and

$$8\pi\epsilon=192\pi^2\eta^2-\frac{16\pi\eta L(8\pi\eta t+a)}{(Lt+M)}+\frac{(L^2-b^2)(8\pi\eta t+a)^2}{4(Lt+M)^2}+\Lambda. \tag{3.9}$$

The model has to satisfy the reality conditions (Ellis 1971, p 117)

(i) $\epsilon + p > 0$

and

(ii) $\epsilon + 3p > 0.$

Assuming $b/L < 1$ we find that if

$$\frac{b^2}{L^2} < \frac{193}{225} \quad \text{and} \quad \zeta < \left\{ 3\frac{b^2}{L^2} - \frac{7}{3} + 3\left[\left(1 - \frac{b^2}{L^2}\right)\left(\frac{130}{144} - \frac{b^2}{L^2}\right) \right]^{1/2} \right\} \eta$$

condition (i) is identically satisfied. Otherwise it leads to the condition

$$\frac{La - 8\pi\eta M}{Lt + M} > P - Q \quad \text{or} \quad \frac{La - 8\pi\eta M}{Lt + M} < -P - Q \tag{3.10}$$

where

$$P = \left[Q^2 - \frac{8L^2}{L^2 - b^2}\left(\frac{67}{3}\eta^2 - 8\frac{b^2}{L^2}\eta^2 - 32\eta\zeta \right) \right]^{1/2}$$

and

$$Q = \frac{8\pi L^2}{L^2 - b^2}\left(\frac{1}{3}\eta - \frac{b^2}{L^2}\eta + \zeta \right).$$

Condition (ii) is identically satisfied if

$$\Lambda < \frac{\pi^2 L^2}{2(L^2 - b^2)}\left(284\frac{b^2}{L^2}\eta^2 + 576\frac{b^2}{L^2}\eta\zeta - 348\eta^2 - 144\zeta^2 - 768\eta\zeta \right).$$

Otherwise we must have

$$\frac{La - 8\pi\eta M}{Lt + M} > R - S \quad \text{or} \quad \frac{La - 8\pi\eta M}{Lt + M} < -R - S \tag{3.11}$$

where

$$R = \frac{2\pi L^2}{L^2 - b^2}\left[87\eta^2 + 36\zeta^2 + 192\eta\zeta - 71\frac{b^2}{L^2}\eta^2 - 144\frac{b^2}{L^2}\eta\zeta + \frac{\Lambda}{2\pi^2}\left(1 - \frac{b^2}{L^2}\right) \right]^{1/2}$$

and

$$S = \frac{4\pi L^2}{L^2 - b^2}\left(4\eta - 2\frac{b^2}{L^2}\eta + 3\zeta \right).$$

The expressions for expansion θ, rotation ω and shear σ_{ij} calculated for the flow vector v^i are given by

$$\theta = \frac{(8\pi\eta t + a)L}{Lt + M} - 24\pi\eta$$
$$\omega = 0 \tag{3.12}$$

and

$$\sigma_{11} = -\frac{L(Lt+M)^{-1}}{3(8\pi\eta t + a)}$$
$$\sigma_{22} = \frac{K^2}{6}\frac{(L+3b)(Lt+M)^{b/L}}{8\pi\eta t + a} \tag{3.13}$$
$$\sigma_{33} = \frac{1}{6K^2}\frac{(L-3b)(Lt+M)^{-b/L}}{8\pi\eta t + a}$$

the other components of the shear tensor σ_{ij} being zero. Hence

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{L^2 + 3b^2}{12}\left(\frac{8\pi\eta t + a}{Lt + M}\right)^2. \tag{3.14}$$

The non-vanishing components of the conformal curvature tensor are

$$C_{12}{}^{12} = C_{13}{}^{13} = -\frac{1}{2}C_{23}{}^{23} = \frac{1}{12}\left(\frac{8\pi\eta t + a}{Lt + M}\right)^2 (L^2 - b^2). \tag{3.15}$$

For large t,

$$C_{23}{}^{23} \simeq -\frac{32}{3}\pi^2\eta^2\left(1 - \frac{b^2}{L^2}\right)$$

and

$$\sigma^2 \simeq \frac{16}{3}\pi^2\eta^2\left(1 + \frac{3b^2}{L^2}\right).$$

Thus the viscosity prevents the free gravitational field as well as the shear from withering away. It is also clear from equation (3.12) that the effect of viscosity is to retard expansion of the model.

4. The second model

The condition $C_{12}{}^{12} = C_{23}{}^{23}$ leads to

$$\left(\frac{A_4}{A}\right)_4 = \frac{C_{44}}{C} - \frac{B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right). \tag{4.1}$$

From equations (2.9), (2.10) and (4.1) we get

$$\frac{B_4}{B} = -8\pi\eta A. \tag{4.2}$$

From equations (2.10) and (4.2) we get

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=2\frac{B_4}{B}\left(\frac{B_4}{B}-\frac{C_4}{C}\right) \tag{4.3}$$

which on integration gives

$$C=B(L-Kt) \tag{4.4}$$

K and L being constants of integration. From equations (4.1) and (4.4) we have

$$\left(\frac{A_4}{A}-\frac{B_4}{B}\right)_4=\left(\frac{A_4}{A}-\frac{B_4}{B}\right)\frac{K}{L-Kt} \tag{4.5}$$

which on integration gives

$$\frac{A_4}{A}-\frac{B_4}{B}=\frac{M}{L-Kt} \tag{4.6}$$

where M is a constant of integration. From equations (4.2) and (4.6) we get

$$A=\left(\frac{8\pi\eta}{M-K}(L-Kt)+N(L-Kt)^{M/K}\right)^{-1} \tag{4.7}$$

N being a constant of integration. From equations (4.2) and (4.7) we obtain

$$B=H\left(\frac{(M-K)N(L-Kt)^{M/K-1}}{8\pi\eta+(M-K)N(L-Kt)^{M/K-1}}\right) \tag{4.8}$$

where H is a constant of integration. Also, from equations (4.4) and (4.8) we obtain

$$C=H\left(\frac{(M-K)N(L-Kt)^{M/K}}{8\pi\eta+(M-K)N(L-Kt)^{M/K-1}}\right). \tag{4.9}$$

By a suitable transformation of coordinates the metric of this model can be put into the form

$$\mathrm{d}s^2=\left(\frac{8\pi\eta}{\alpha-1}T+\beta T^{\alpha}\right)^{-2}\left(\mathrm{d}X^2-\mathrm{d}T^2+T^{2\alpha}\,\mathrm{d}Y^2+T^{2\alpha+2}\,\mathrm{d}Z^2\right) \tag{4.10}$$

where α and β are arbitrary constants. The distribution in the model is given by

$$8\pi p=64\pi^2\eta^2\frac{2-\alpha}{\alpha-1}+16\pi\eta\alpha\beta T^{\alpha-1}+\alpha(\alpha-1)\beta^2T^{2(\alpha-1)}-\left(\frac{8\pi\eta}{\alpha-1}+\alpha\beta T^{\alpha-1}\right)^2$$
$$-\frac{16\pi\eta}{3}(\alpha+2)\left(\frac{8\pi\eta}{\alpha-1}+\beta T^{\alpha-1}\right)+8\pi\zeta[(\alpha-1)\beta T^{\alpha-1}-16\pi\eta]-\Lambda \tag{4.11}$$

and

$$8\pi\epsilon=64\pi^2\eta^2\left(1-\frac{\alpha}{(\alpha-1)^2}\right)+16\pi\eta\beta T^{\alpha-1}\left(1-\frac{\alpha^2}{\alpha-1}\right)-\alpha\beta^2T^{2(\alpha-1)}+\Lambda. \tag{4.12}$$

The reality conditions impose restrictions on the time during which the model exists.

The expression for the expansion θ for the flow vector V^i is given by

$$\theta=\beta(\alpha-1)T^{\alpha-1}-16\pi\eta. \tag{4.13}$$

The rotation ω is identically zero and the magnitude of the shear is given by

$$\sigma^2 = \frac{1}{3}(\alpha^2+\alpha+1)\left(\frac{8\pi\eta}{\alpha-1}+\beta T^{\alpha-1}\right)^2. \tag{4.14}$$

The non-vanishing components of the conformal curvature tensor are

$$C_{12}{}^{12} = C_{23}{}^{23} = -\frac{1}{2}C_{13}{}^{13} = -\frac{1}{3}\alpha\left(\frac{8\pi\eta}{\alpha-1}+\beta T^{\alpha-1}\right). \tag{4.15}$$

Thus, for this model too, the effect of viscosity is to prevent the shear and the free gravitational field from withering away for large values of T. It also retards the expansion of the model.

The metric (4.10) is conformal to the metric

$$ds^2 = dX^2 - dT^2 + T^{2\alpha}\,dY^2 + T^{2\alpha+2}\,dZ^2. \tag{4.16}$$

Equation (4.16) represents a viscous fluid cosmological model in which the kinematic viscosity η_0 is $-\alpha/8\pi T$ and the pressure p_0 and density ϵ_0 are given by

$$8\pi p_0 = 8\pi\zeta\left(\frac{2\alpha+1}{T}\right) - \frac{5\alpha^2+\alpha}{3T^2} - \Lambda \tag{4.17}$$

and

$$8\pi\epsilon_0 = \frac{\alpha(\alpha+1)}{T^2} + \Lambda. \tag{4.18}$$

Reality conditions require that

$$-\tfrac{1}{2} < \alpha < 0$$

and that ζ is greater than the greater of

$$\frac{\alpha(\alpha-1)}{12\pi T(2\alpha+1)} \quad \text{and} \quad \frac{1}{12\pi(2\alpha+1)}\left(\frac{2\alpha^2}{T}+\Lambda T\right). \tag{4.19}$$

It is also to be noted that the space–time equation (4.16) becomes flat when α is zero. The corresponding model

$$ds^2 = (\beta - 8\pi\eta T)^{-2}(dX^2 - dT^2 + dY^2 + T^2\,dZ^2) \tag{4.20}$$

represents a conformally flat viscous fluid cosmological model.

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